Advanced Algebra I

Text Book:

Topices in Algebra (Second edition) Wiley Eastern limited – I.N. Herstein.

Book for reference:

- A Course in Abstract Algebra (3rd edition) – Vijay K. Khanna, S.K. Bhambri – viskas publishing House New delhi
- Fields and rings – kaplemsky, Irving (2nd edition) – University of Chicago – Chicago (1972)

Unit – I

vector space: Dual spaces – Inner product spaces

sec: 4.3 and 4.4

Unit – II

Linear transformations: The Algebra of linear transformations – Characteristic roots – Matrices.

sec: 6.1 – 6.3

Unit – III

canonical Forms: Triangular form – Nilpotent form – Jordan form

sec: 6.4 – 6.6

unit – IV

Matrices: Trace and transpose – Determinats.

sec: 6.8 – 6.9

unit – V

Transformations: Hermitian, unitary and normal transformations

sec: 6.10 (upto Lemma 6.10.11)

Unit - I

Vector space

Defn: 4.1

Let F be a given field and let V be a non-empty set with rules of addition and scalar Multiplication which assigns to any $a, b \in V$, a sum $a+b \in V$ and to any $a \in V$, $\alpha \in F$ a product $\alpha a \in V$.

Then V is called a vector space over F if the following axioms hold

(i) $\alpha(v+w) = \alpha v + \alpha w$

(ii) $(\alpha+\beta)v = \alpha v + \beta v$

(iii) $\alpha(\beta v) = (\alpha\beta)v$

(iv) $1 \cdot v = v$, $\forall \alpha, \beta \in F$, $v, w \in V$.

[1 represent the unit element of F under multiplication]

Example: 4.1.1

Let F be a field and let V be the totally of all n-tuples $(\alpha_1, \alpha_2, \ldots, \alpha_n)$ where the $\alpha_i \in F$.

Two elements $(\alpha_1, \alpha_2, \ldots, \alpha_n)$ and $(\beta_1, \beta_2, \ldots, \beta_n) \in V$ iff $\alpha_i = \beta_i$ for each $i = 1, 2, \ldots, n$.

(i) $(\alpha_1, \alpha_2, \ldots, \alpha_n) + (\beta_1, \beta_2, \ldots, \beta_n) = (\alpha_1+\beta_1, \alpha_2+\beta_2, \ldots, \alpha_n+\beta_n)$

(ii) $\gamma(\alpha_1, \alpha_2, \ldots, \alpha_n) = (\gamma\alpha_1, \gamma\alpha_2, \ldots, \gamma\alpha_n)$, for $\gamma \in F$

$\therefore$ V is a vector space over F.

Example: 4.1.2

Let F be any field and let $V = F[x]$, the set of polynomials in x over F.

Then $F[x]$ is a vector space over F.

Example: 4.1.3

In $F[x]$. Let V_n be the set of all polynomials of degree less than n.

Then V_n is a vector space over F.

Subspace

Defn:

If V is a vector space over F and if $W \subset V$ then W is a subspace of V if under the operations of V, W itself equivalently W is a subspace of V whenever $w_1, w_2 \in W$, $\alpha, \beta \in F \Rightarrow \alpha w_1 + \beta w_2 \in W$.

Defn: If U and V are vector space over F then the mapping T of U into V ($T: U \to V$) is said to be a homomorphism if (i) $(u_1+u_2)T = u_1T + u_2T$ (ii) $(\alpha u_1)T = \alpha(u_1T)$ $\forall u_1, u_2 \in U$, $\forall \alpha \in F$

Note: If T is one-one then T is called an isomorphism

Defn: Kernal of T.

The Kernal of T as defined as $\{u \in U / uT = 0\}$, where 0 is the identity element of the addition in V.

Note: 1) The kernal of T is a subspace of U and that T is an isomorphism iff its Kernal is zero.

2) Two vectors space are said to be isomorphic if there is an isomorphism of one onto the other.

3) The set of all homomorphism of U into V will be written as $Hom(U, V)$.

Lemma: 4.1.1

If V is a vector space over F.

i) $\alpha . 0 = 0$ for $\alpha \in F$

(ii) $0 . v = 0$ for $v \in V$.

(iii) $(-\alpha) v = -(\alpha v)$ for $\alpha \in F, v \in V$.

(iv) if $v \neq 0$, then $\alpha v = 0 \Rightarrow \alpha = 0$.

Lemma: 4.1.2

If V is a vector space over F and if W is a subspace of V then V/W is a vector space of F,

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where v_1+w, $v_2+w \in V/W$ and $\alpha \in F$.

(i) $(v_1+w)+(v_2+w) = (v_1+v_2)+w$.

(ii) $\alpha(v_1+w) = \alpha v_1 + w$, V/W is called the quotient space of V by W.

Theorem: 4.1.1.

If T is a homomorphism of U onto V with kernal W then V is isomorphic to U/W.

Conversely, if U is a vector space and W is subspace of U then there is a homomorphism of U onto U/W.

Let V be a vector space over F and let $u_1, u_2, \ldots, u_n$ be subspace of V. V is said to be the internal direct sum $v_1, v_2, \ldots, v_n$ if every element $v \in V$ can be written in one and only $v = u_1+u_2+\cdots+u_n$ where $u_i \in U_i$.

Theorem: 4·1·2

If V is the internal direct sum of $u_1, u_2, \ldots, u_n$ then V is isomorphic to a extended direct sum of $u_1, u_2, \ldots, u_n$.

4·2 Linear Independence and Basis

Defn:

*If V is a vector space over F and if $v_1, v_2, \ldots, v_n \in V$ then any element of the form $\alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_n v_n$, where $\alpha_i \in F$ is a linear combination over F of $v_1, v_2, \ldots, v_n$.

Defn: * If S is a non-empty subset of the vector space V, then $L(S)$, the linear span of S, is the se[t] of all linear combinations of finite sets of elemen[ts]

Lemma 4.2.1

$L(S)$ is a subspace of V.

Lemma 4.2.2

If S, T are subset of V. then

(i) $S \subset T \Rightarrow L(S) \subset L(T)$

(ii) $L(S \cup T) = L(S) + L(T)$

(iii) $L(L(S)) = L(S)$.

Defn: The vector space V is said to be finite dimensional if there is a finite subset S in V such that $V = L(S)$.

Defn: Linearly dependent

If V is a vector space and if $v_1, v_2, \ldots, v_n$ are in V, we say that they are linearly dependent over F if $\exists$ elements $\lambda_1, \lambda_2, \ldots, \lambda_n$ in F not all of them zero, such that $\lambda_1 v_1 + \lambda_2 v_2 + \ldots + \lambda_n v_n = 0$.

Lemma: 4.2.3

If $v_1, v_2, \ldots, v_n \in V$ are linearly independent then every element in their linear span has a unique representation in the form $\lambda_1 v_1 + \lambda_2 v_2 + \ldots + \lambda_n v_n$ with the $\lambda_i \in F$.

Theorem: 4.2.1

If $v_1, v_2, \ldots, v_n$ are in V then either they are linearly independent or some v_k is a linear combination of the preceding ones, $v_1, v_2, \ldots, v_{k-1}$.

Note:

1) If $v_1, v_2, \ldots, v_n$ in V have W as linear span and if $v_1, v_2, \ldots, v_k$ are linearly independent then we can find a subset of $v_1, v_2, \ldots, v_n$ of the form $v_1, v_2, \ldots, v_k, v_{i_1}, \ldots, v_{i_r}$ consisting of linearly

independent elements whose linear span is also W.

2) If V is a finite-dimensional vector space then it contains a finite set $v_1, v_2, \ldots, v_n$ of linearly independent elements whose linear span is V.

Defn: A subset S of a vector space V is called a basis of V if S consists of linearly independent elements and $V = L(S)$.

Note: 3) If V is a finite dimensional vector space and if $u_1, u_2, \ldots, u_m$ span V then some subset of $u_1, u_2, \ldots, u_m$ forms a basis of V.

Lemma: 4.2.4

If $v_1, v_2, \ldots, v_n$ is a basis of V over F and if $w_1, w_2, \ldots, w_m$ in V are linearly independent over F then $m \leq n$.

Note:

(1) If V is finite dimensional over F then any two basis of V have the same no. of elements.

(2) $F^{(n)}$ is isomorphic $F^{(m)}$ iff $n = m$.

(3) If V is finite dimensional over F then V is isomorphic to $F^{(n)}$ for a unique integer n; in fact n is the no. of elements in any basis of V over F.

(4) Any two finite dimensional vector space over F of the same dimension are isomorphic.

Lemma: 4.2.5

If V is finite dimensional over F and if $u_1, u_2, \ldots, u_m \in V$ are linearly independent then we can find vectors $u_{m+1}, u_{m+2}, \ldots, u_{m+r}$ in V such that $u_1, u_2, \ldots, u_m, u_{m+1}, \ldots, u_{m+r}$ is a basis of V

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Lemma: 4·2·6

If V is finite dimensional and if W is a subspace of V then W is finite dimensional $\dim W \leq \dim V$ and $\dim V/W = \dim V - \dim W$.

If A and B are finite-dimensional subspace of a vector space V, then $A+B$ is finite-dimensional and $\dim(A+B) = \dim(A) + \dim(B) - \dim(A \cap B)$.

Unit – I

Sec: 4·3

Dual spaces

(Field → F is a nonzero commutative ring & abelian gp)

J (*) 2009 Lemma: 4.3.1 ($(F, +, \cdot)$ satisfy axioms closure, asso. commutative, identity, inverse addition & multiplication, additive & multiplication)

$Hom(V, W)$ is a vector space over F.

[Vector space: A nonempty V is said to be vector space over a field F. If V is an abelian group under an operation we denoted by $\oplus$ and if every $\alpha \in F$, $v \in V$. there is defined an element αv in V subject to.

(i) $\alpha(v+w) = \alpha v + \alpha w$

(ii) $(\alpha+\beta)v = \alpha v + \beta v$

(iii) $\alpha(\beta v) = (\alpha\beta)v$

(iv) $1 . v = v \quad \forall \alpha, \beta \in F$ and $v, w \in V$.

Where 1 represented the unit element of under multiplication

Proof:

Let $S, T \in Hom(V, W)$

T.P $S+T \in Hom(V, W)$

Then S and T are vector space Homomorphism of V into W.

ie) $(v_1 + v_2)S = v_1 S + v_2 S$

$(\alpha v_1)S = \alpha(v_1 S)$

$(v_1 + v_2)T = v_1 T + v_2 T$

$(\alpha v_1)T = \alpha(v_1 T) \quad \forall v_1, v_2 \in V$ and $\alpha \in F$

<u>To Prove</u>: $Hom(V, W)$ is an abelian group under $\oplus$

Define $S+T : V \to W$ by $v(S+T) = vS + vT$

It is enough to prove that $S+T \in Hom(V,W)$

(i) $(v_1+v_2)(S+T) = (v_1+v_2)S + (v_1+v_2)T$

$= v_1 S + v_2 S + v_1 T + v_2 T$

$= v_1 S + v_1 T + v_2 S + v_2 T$

$= v_1(S+T) + v_2(S+T)$

$\therefore (v_1+v_2)(S+T) = v_1(S+T) + v_2(S+T)$ —— ①

ii) $(\alpha v_1)(S+T) = (\alpha v_1)S + (\alpha v_1)T$

$= \alpha(v_1 S) + \alpha(v_1 T)$

$= \alpha(v_1 S + v_1 T)$

$\therefore (\alpha v_1)(S+T) = \alpha(v_1 S + v_1 T)$ ~~$\alpha v_1(S+T)$~~ —— ②

Eqn ①, ② is true for all $v_1, v_2 \in V$ and $\alpha \in F$.

$\therefore S+T \in Hom(V,W)$

Cleary '+' is associative & commutative

Let 0 be the Homomorphism of V into W.

$0 : V \to W$ by $0(v) = 0$, [where 0 is the 0 element and W is the identity in $Hom(V,W)$]

Every element of V onto the zero element of W.

Let $S \in Hom(V,W)$.

Then $-S$ defined by $v(-S) = -(vS)$.

$(v_1+v_2)(-S) = -((v_1+v_2)S)$

$= -(v_1 S + v_2 S)$

$= -(v_1 S) + (-(v_2 S))$

$= v_1(-S) + v_2(-S)$

$(v_1+v_2)(-S) = v_1(-S) + v_2(-S)$

$$(\alpha v_1)(-s) = -((\alpha v_1)s)$$
$$= -(\alpha(v_1 s))$$
$$= \alpha(v_1(-s))$$
$$(\alpha v_1)(-s) = \alpha(v_1(-s))$$

$\in Hom(V,W)$

$\therefore$ $Hom(V,W)$ is an abelian group under the addition.

For $\lambda \in F$ and ~~Ko~~ $S \in Hom(V,W)$.

Define: λS by $v(\lambda S) = \lambda(vS)$ $\forall$ $v \in V$.

Claim: $\lambda S \in Hom(V,W)$

$$(v_1 + v_2)\lambda S = \lambda((v_1+v_2)S)$$
$$= \lambda(v_1 S + v_2 S)$$
$$= \lambda(v_1 S) + \lambda(v_2 S)$$
$$= v_1(\lambda S) + v_2(\lambda S)$$
$$(v_1 + v_2)\lambda S = v_1(\lambda S) + v_2(\lambda S)$$

$$(\alpha v_1)\lambda S = \lambda(\alpha v_1 S)$$
$$= \lambda\alpha(v_1 S)$$
$$= \alpha\lambda(v_1 S)$$
$$(\alpha v_1)\lambda S = \alpha(v_1 \lambda S)$$

$\therefore$ $\lambda S \in Hom(V,W)$.

(i) Let $\alpha \in F$ and $S, T \in Hom(V,W)$

~~Define~~ To prove: $\alpha(T+S) = \alpha T + \alpha S$

$$V(\alpha(T+S)) = \alpha(V(T+S))$$
$$= \alpha(VT + VS)$$
$$= \alpha(VT) + \alpha(VS)$$
$$= V(\alpha T) + V(\alpha S)$$

$= v(\alpha T + \alpha S)$ is true $\forall v \in V$.

$\therefore \alpha(T+S) = \alpha T + \alpha S$

(ii) Let $\alpha, \beta \in F$ and $T \in Hom(V,W)$

To prove: $(\alpha+\beta)T = \alpha T + \beta T$.

$$v((\alpha+\beta)T) = (\alpha+\beta)(vT)$$
$$= \alpha(vT) + \beta(vT)$$
$$= v(\alpha T) + v(\beta T)$$
$$= v(\alpha T + \beta T) \text{ is true } \forall v \in V.$$

$(\alpha+\beta)T = \alpha T + \beta T$.

iii) claim: $\alpha(\beta T) = (\alpha\beta)T$.

$$v(\alpha(\beta T)) = \alpha(v(\beta T))$$
$$= (\alpha\beta)v(T)$$
$$= v((\alpha\beta)(T)) \text{ is true } \forall v \in V.$$

$\Rightarrow \alpha(\beta T) = (\alpha\beta)T$.

Clearly $1.T = T$.

$\therefore Hom(V,W)$ is a vector space over F.

Theorem: 4.3.1

2009 J If V and W are of dimensions ~~of~~ m and n respectively over F then $Hom(V,W)$ is of dimension mn over F.

Proof: We shall prove the theorem by exihibit a basis of $Hom(V,W)$ over F consisting of mn elements.

Let $v_1, v_2, \dots, v_m$ be a basis ~~of~~ V over F and $w_1, w_2, \dots, w_n$ be a basis of W over F.

If $v \in V$ then $v = \lambda_1 v_1 + \lambda_2 v_2 + \dots + \lambda_m v_m$ where $\lambda_1, \lambda_2, \dots, \lambda_m$ are uniquely determined elements of F.

Define $T_{ij} : V \to W$ by $vT_{ij} = \lambda_i w_j$

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From the point of view of bases

Let $v_k T_{ij} = 0$ for $k \neq i$ and $v_i T_{ij} = w_j$

<u>Claim:</u> $T_{ij} \in Hom(V, W)$

addition & scalar multiplication

Let $v, v' \in V$.

Then $v = \lambda_1 v_1 + \lambda_2 v_2 + \cdots + \lambda_m v_m$

$v' = \alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_m v_m$, $\alpha_i, \lambda_i \in F$.

Then $v + v' = (\lambda_1 + \alpha_1) v_1 + \cdots + (\lambda_m + \alpha_m) v_m$

$(v + v') T_{ij} = (\lambda_i + \alpha_i) w_j$

$= \lambda_i w_j + \alpha_i w_j$

$= v T_{ij} + v' T_{ij}$

Hom(V,W) map define addition scalar mult

$(\alpha+\beta)w = \alpha w + \beta w$

$(\alpha\beta)w = \alpha(\beta w)$

$\alpha v = \alpha \lambda_1 v_1 + \alpha \lambda_2 v_2 + \cdots + \alpha \lambda_m v_m$.

$(\alpha v) T_{ij} = \alpha \lambda_i w_j$

$= \alpha(\lambda_i w_j)$

$= \alpha(v T_{ij})$

$\Rightarrow T_{ij} \in Hom(V, W)$.

Since i varies from 1 to m and j varies from 1 to n. then there are mn such that T_{ij}'s

<u>**Claim:**</u> $\{T_{11}, T_{12}, \ldots, T_{1n}, T_{21}, T_{22}, \ldots T_{2n}, \ldots, T_{m1}, T_{m2}, \ldots T_{mn}\}$ span $Hom(V, W)$ over F

~~Constitute~~ ~~a basis of Hom(V, W) over F.~~

Let $S \in Hom(V, W)$

Then $v_1 S \in W$.

$\Rightarrow v_1 S = \alpha_{11} w_1 + \alpha_{12} w_2 + \cdots + \alpha_{1n} w_n$

where $\alpha_{11}, \alpha_{12}, \ldots \alpha_{1n} \in F$.

Similiarly, $v_i S = \alpha_{i1} w_1 + \alpha_{i2} w_2 + \cdots + \alpha_{in} w_n$

For $i = 1$ to m

Consider,

$$S_0 = \alpha_{11} T_{11} + \alpha_{12} T_{12} + \cdots + \alpha_{1n} T_{1n} + \alpha_{21} T_{21} + \cdots + \alpha_{2n} T_{2n} + \cdots + \alpha_{m1} T_{m1} + \cdots + \alpha_{mn} T_{mn}$$

Now,

$$v_k S_0 = v_k (\alpha_{11} T_{11} + \alpha_{12} T_{12} + \cdots + \alpha_{mn} T_{mn})$$

$$= \alpha_{11}(v_k T_{11}) + \alpha_{12}(v_k T_{12}) + \cdots + \alpha_{mn}(v_k T_{mn})$$

now $v_k S_0 = \alpha_{k1} w_1 + \alpha_{k2} w_2 + \cdots + \alpha_{kn} w_n$

$= v_k S$ is true $\forall$ basis of V.

$\Rightarrow S_0 = S$

$\Rightarrow S$ is a linear combination of mn such T_{ij}'s

Hence the mn elements $T_{11}, T_{12}, \ldots, T_{1n}, T_{21}, T_{22}, \ldots, T_{2n}, \ldots, T_{m1}, T_{m2}, \ldots \ldots T_{mn}$ span Hom (V, W) over F.

<u>To prove</u>: $\{T_{11}, T_{12}, T_{13}, \ldots, T_{mn}\}$ is a linearly independent over F.

Suppose that $\beta_{11} T_{11} + \beta_{12} T_{12} + \cdots + \beta_{1n} T_{1n} + \cdots + \beta_{m1} T_{m1} + \cdots + \beta_{mn} T_{mn} = 0$ where β_{ij}'s are in F.

$$\Rightarrow v_k (\beta_{11} T_{11} + \beta_{12} T_{12} + \cdots + \beta_{mn} T_{mn}) = v_k(0)$$

$$\Rightarrow \beta_{11}(v_k T_{11}) + \beta_{12}(v_k T_{12}) + \cdots + \beta_{k1}(v_k T_{k1}) + \beta_{k2}(v_k T_{k2}) + \cdots + \beta_{kn}(v_k T_{kn}) + \cdots + \beta_{mn}(v_k T_{mn}) = 0$$

$$\Rightarrow \beta_{k1} w_1 + \beta_{k2} w_2 + \cdots + \beta_{kn} w_n = 0.$$

Since $\{w_1, w_2, \ldots, w_n\}$ are linearly independent then $\beta_{k1} = \beta_{k2} = \cdots = \beta_{kn} = 0$.

Similiarly, apply $v_1, v_2, \ldots, v_m$ we get

$$\beta_{11} = \cdots = \beta_{1n} = \beta_{21} = \cdots = \beta_{m1} = \cdots = \beta_{mn} = 0$$

Hence the mn T_{ij}'s from a basis of Hom (V, W) over F

Hence $\dim(\mathrm{Hom}(V, W)) = mn$

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Corollary: 1

If $\dim_F V = m$ then $\dim_F \operatorname{Hom}(V, V) = m^2$

By the above Theorem. Put $V = m$ then

$\dim_F \operatorname{Hom}(V, V) = m^2$

corollary: 2

If $\dim_F V = m$ then $\dim_F \operatorname{Hom}(V, F) = m$

Since a vector space F is of dimension 1 over F.

ie) $\dim_F F = 1$ then by the above Theorem.

$\dim_F \operatorname{Hom}(V, F) = m \cdot 1 = m$.

Note: If V is finite dimensional over F then V is isomorphic to $\operatorname{Hom}(V, F)$.

Defn: The dual space of a vector space V is $\operatorname{Hom}(V,F)$ and it is denoted by $\hat{V}$.

Defn: An element of $\hat{V}$ is called linear functional on V into F ie) the element of $\hat{V}$ are functions defined on V and having their values in F.

Lemma: 4.3.2

If V is finite-dimensional and $v \neq 0 \in V$, then there is an element $f \in \hat{V}$ such that $f(v) \neq 0$.

Proof:

If V is finite-dimensional vector space over F and let $v_1, v_2, \ldots, v_n$ be a basis of V.

Let $\hat{v}_i$ be the element of $\hat{V}$ defined by

$$v_i(v_j) = \begin{cases} 0 & \text{for } i \neq j \\ 1 & \text{if } i = j \end{cases}$$

If $v \in V$ then $v = \alpha_1 v_1 + \alpha_2 v_2 + \dots + \alpha_n v_n$

Then $\hat{v}_i(v) = \hat{v}_i(\alpha_1 v_1 + \alpha_2 v_2 + \dots + \alpha_i v_i + \dots + \alpha_n v_n)$

$= \alpha_i$

$\cancel{= T_{ij}(v)}$

Take $W = F$ is one dimensional over F.

W are of dimensions m respectively over F then of dimension over F

Then $j = 1$.

~~By Theorem: 4.3.1~~, w.k.t $\hat{v}_1, \hat{v}_2, \dots, \hat{v}_n$ ~~from~~ form a basis of $\hat{V}$ this basis is called the dual basis of $v_1, v_2, \dots, v_n$.

Find a basis $v_1 = v, v_2, \dots, v_n$ and

So there is an element in $\hat{V}$ namely $\hat{v}_1$

~~Take~~ $\hat{v}_1 = \hat{v}$

& ~~Now~~, $\hat{v}_1(v_1) = \hat{v}_1(v) = 1 \neq 0$.

Hence there is an element $\hat{v}_1 \in \hat{V}$ such that $\hat{v}_1(v) \neq 0$.

Note:

If $v_0 \in V$ and $f \in \hat{V}$ then $f(v_0)$ defines a functional on $\hat{V}$ into F. Let us denote this function by T_{v_0}.

ie) $T_{v_0}(f) = f(v_0)$ for any $f \in \hat{V}$.

T_{v_0} is a homomorphism of $\hat{V}$ to F.

Proof: Let $f, g \in V$.

Proof vendam.

Claim: $T_{v_0}(f+g) = (f+g)v_0)$

$= f(v_0) + g(v_0)$

$= T_{v_0}(f) + T_{v_0}(g)$.

$T_{v_0}(\lambda f) = ((\lambda f)v_0)$

$= \lambda(f(v_0))$

$= \lambda(T_{v_0}(f))$

$\Rightarrow T_{v_0} \in$ dual space of $\hat{V} = \hat{\hat{V}}$.

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Lemma: 4.3.3

If V is finite-dimensional, then ψ is an isomorphism of V onto $\hat{\hat{V}}$.

Proof:

T_v

Given any element $v \in V$. we can associate with it an element $\underline{T_v}$ in $\hat{\hat{V}}$

Define $\psi : V \to \hat{\hat{V}}$ by $v\psi = T_v$.

<u>Claim</u>: ψ is an homomorphism.

<u>To prove</u>: $(v_1 + v_2)\psi = v_1\psi + v_2\psi$

$$(v_1 + v_2)\psi = T_{v_1} + T_{v_2} \quad \text{—— (1)}$$

Now, $T_{v_1+v_2}(f) = f(v_1 + v_2)$

$= f(v_1) + f(v_2)$

$= T_{v_1}(f) + T_{v_2}(f)$

$= (T_{v_1} + T_{v_2})(f)$

$\Rightarrow T_{v_1+v_2} = T_{v_1} + T_{v_2}$.

(1) $\Rightarrow (v_1 + v_2)\psi = T_{v_1} + T_{v_2}$

$= v_1\psi + v_2\psi$.

<u>To prove</u>: $(\lambda v_1)\psi = \lambda(v_1\psi)$

$$(\lambda v_1)\psi = T_{\lambda v_1} \quad \text{—— (2)}$$

Now, $T_{\lambda v_1}(f) = f(\lambda v_1)$

$= \lambda(f(v_1))$

$= \lambda(T_{v_1}(f))$

$= (\lambda T_{v_1})(f)$

$\Rightarrow T_{\lambda v_1} = \lambda T_{v_1}$

(2) $\Rightarrow (\lambda v_1)\psi = T_{\lambda v_1} = \lambda T_{v_1} = \lambda(v_1\psi)$

$\therefore \psi$ is an homomorphism.

claim: ψ is 1-1

To prove: $\ker \psi = \{0\}$

Let $v \in \ker \psi$

$v\psi = 0$.

$\Rightarrow T_v(f) = 0, \ \forall f \in \hat{V}$.

$f(v) = 0, \forall f \in \hat{V}$

By Lemma: 4.3.2, $v = 0$.

If V is a finite dimensional and $v \neq 0 \in V$ then there is an element $f \in \hat{V}$ such that $f(v) \neq 0$

$\therefore \ker \psi = 0 \Rightarrow \psi$ is 1-1

$\therefore \psi$ is an isomorphism.

By Corollary: 2 of the Theorem: 4.3.1, If $\dim_F V = m$ then $\dim_F Hom(V, F) = m$

V and $\hat{V}$ has same dimension.

Similarly $\hat{V}$ and $\hat{\hat{V}}$ have same dimension.

Thus ψ must be onto.

Defn: If W is a subspace of V then the annihilator of W, $A(W) = \{f \in \hat{V} / f(w) = 0 \ \forall w \in W\}$.

Note:

1) $A(W)$ is a subspace of $\hat{V}$.

Proof: Let $f, g \in A(W)$ and $\alpha, \beta \in F$.

Then $f, g \in \hat{V}$ such that $f(w) = 0$ and $g(w) = 0 \ \forall w \in W$

Now, $(\alpha f + \beta g)w = (\alpha f)(w) + (\beta g)w$

$= \alpha(f w) + \beta(g w)$

$= 0$.

$\therefore (\alpha f + \beta g)(w) = 0 \ \forall w \in W$.

$\alpha f + \beta g \in A(W)$.

$\therefore A(W)$ is a subspace of $\hat{V}$.

2) If $U \subset W$ then $A(U) \supset A(W)$

Let $f \in A(W)$ then $f(w) = 0 \ \forall w \in W$.

$\Rightarrow f(u) = 0 \ \forall u \in U \ (\because U \subset W)$

$\Rightarrow f \in A(U)$.

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Theorem: 4.3.2

If V is finite-dimensional and W is a subspace of V, then $\hat{W}$ is isomorphic to $\hat{V}/A(W)$ and $\dim A(W) = \dim V - \dim W$.

Proof:

Let W be a subspace of V, where V is finite-dimensional.

If $f \in \hat{V}$, then the restriction of f to W is denoted by $\tilde{f}$ and is defined by $\tilde{f}(w) = f(w)$ for every $w \in W$.

Since $f \in \hat{V}$, then $\tilde{f} \in \hat{W}$

Define a map $T: \hat{V} \to \hat{W}$ by $fT = \tilde{f}$ for $f \in \hat{V}$

Claim: T is a homomorphism.

To Prove: $(f+g)T = fT + gT$.

$$(f+g)T = \widetilde{(f+g)}$$

$$\widetilde{(f+g)}(w_1) = (f+g)(w_1)$$
$$= f(w_1) + g(w_1)$$
$$= \tilde{f}(w_1) + \tilde{g}(w_1)$$
$$= (\tilde{f} + \tilde{g})(w_1).$$

$$\Rightarrow \widetilde{(f+g)} = \tilde{f} + \tilde{g}.$$

$$\therefore (f+g)T = \widetilde{(f+g)} = \tilde{f} + \tilde{g}.$$

$$\therefore (f+g)T = fT + gT.$$

To Prove: $(\lambda f)T = \lambda(f(T))$

$$\widetilde{\lambda f}(w_1) = \lambda f(w_1)$$
$$= \lambda(f(w_1))$$
$$= \lambda \tilde{f}(w_1)$$

$$\therefore \widetilde{\lambda f} = \lambda \tilde{f}$$

$$\therefore (\lambda f)T = \widetilde{\lambda f} = \lambda \tilde{f} = \lambda(f(T)).$$

$\therefore$ T is a homomorphism.

<u>To prove:</u> $\ker T = A(W)$.

$f \in \ker T$

$fT = 0$

$\Rightarrow \tilde{f} = 0$.

$\Rightarrow \tilde{f}(w) = 0 \ \forall w \in W$.

$\Rightarrow f(w) = 0 \ \forall w \in W$.

Also $f \in \hat{V}$ {by Annihilator defn}

Hence $f \in A(W)$.

Conversely if $f \in A(W)$

$\Rightarrow f \in \hat{V}$ and $f(w) = 0 \ \forall w \in W$.

$\Rightarrow \tilde{f}(w) = 0 \ \forall w \in W$.

$\Rightarrow \tilde{f} = 0$

$\Rightarrow fT = 0$.

$f \in \ker T$.

$\therefore \ker T = A(W)$.

<u>Claim:</u> T is onto.

It, ETPT ~~If~~ ~~let~~ $h \in \hat{W}$.

Then h is the restriction of some $f \in \hat{V}$.

ie) $h = \tilde{f}$ for some $f \in \hat{V}$.

Lemma 4.2.5 If ~~Let~~ $w_1, w_2, \ldots, w_m$ ~~be~~ a basis of W. then it can be expanded to a basis of V of the form

Since $\{w_1, w_2, \ldots, w_m\}$ are linearly independent.

~~Then by Lemma: 4.2.5, we can find vectors~~ ~~$v_1, v_2, \ldots, v_r$ such that~~ $w_1, w_2, \ldots, w_m, v_1, v_2, \ldots, v_r$ ~~are basis of V.~~ where

$\Rightarrow r + m = \dim V$.

Let W_1 be the subspace of V spanned by $v_1, v_2, \ldots, v_r$.

Thus $V = W \oplus W_1$

If $v \in V$ then $v = w + w_1$ where $w \in W$ and $w_1 \in W_1$

Define $f : V \to F$ by $f(v) = f(w + w_1) = h(w)$.

<u>Claim:</u> f is a homomorphism.

Let $v_1, v_2 \in V \Rightarrow v_1 = w + w_1$ and $v_2 = w' + w_2$

where $w, w' \in W$ and $w_1, w_2 \in W_1$.

<u>To Prove:</u> $f(v_1 + v_2) = f(v_1) + f(v_2)$

Now, $f(v_1 + v_2) = h(w + w') \quad (\because h \in \hat{W})$

$= h(w) + h(w')$

$= f(v_1) + f(v_2)$

<u>To Prove:</u> $f(\alpha v_1) = \alpha f(v_1)$

$f(\alpha v_1) = f(\alpha w + \alpha w_1)$

$= h(\alpha w)$

$= \alpha h(w)$

$= \alpha f(v_1)$

Hence f is a homomorphism.

$\Rightarrow f \in \hat{V}$

$\therefore$ for all $h \in \hat{W}$, $\exists f \in \hat{V}$ such that

fT = restriction of $f = \tilde{f} = h$.

$\therefore T$ is onto.

By Theorem: 4.1.1 [If T is a homomorphism of U onto V and $\ker T = W$ then $V \cong \frac{U}{W}$].

ie) $\hat{W}$ and $\frac{\hat{V}}{A(W)}$ have same dimension.

ie) $\dim(\hat{W}) = \dim\left(\hat{V}/A(W)\right)$ — ①

By corollary: 2 of Theorem: 4.3.1, If $\dim_F V = m$ then $\dim_F \mathrm{Hom}(V, F) =$

$\dim W = \dim \hat{W}$ and $\dim V = \dim \hat{V}$. — ②

By Lemma: 4.2.6,

$$\dim\left(\frac{\hat{V}}{A(W)}\right) = \dim(\hat{V}) - \dim(A(W))$$

From ① and ②

$$\dim(\hat{W}) = \dim(\hat{V}) - \dim(A(W))$$

$$\dim(W) = \dim(V) - \dim(A(W))$$

$$\Rightarrow \dim(A(W)) = \dim V - \dim W.$$

Corollary:

$$A(A(W)) = W.$$

Proof: <u>Claim</u>: $W \subseteq A(A(W))$

Let $w \in W$.

Define a map ψ by $w\psi = T_w$ where $T_w(f) = f(w)$

If $f \in A(W)$

$\Rightarrow f(w) = 0 \quad \forall w \in W.$

$\Rightarrow T_w(f) = 0 \quad \forall f \in A(W).$

$\Rightarrow T_w \in A(A(W))$

$\Rightarrow W \subseteq A(A(W))$ —— ⊛.

$\dim A(A(W)) = \dim \hat{\hat{V}} - \dim A(W).$ [$\because A(W) \subset \hat{V}$ by using the above Thm]

$= \dim V - [\dim V - \dim W]$ [$\because \dim \hat{\hat{V}} = \dim V$, $\dim A(W) = \dim V - \dim W$]

$= \dim V - \dim V + \dim W.$

From ① & ⊛ $\dim A(A(W)) = \dim W$.

$A(A(W)) = W.$

2011. Theorem: 4.3.3

J AP-2019 If the system of homogeneous linear equations:

$$a_{11}x_1 + \dots + a_{1n}x_n = 0,$$

$$a_{21}x_1 + \dots + a_{2n}x_n = 0,$$

$$\vdots$$

$$a_{m1}x_1 + \dots + a_{mn}x_n = 0,$$ where $a_{ij} \in F$ is of rank r, then there are $n-r$ linearly independent solutions in $F^{(n)}$.

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Proof:

If $F^{(n)}$ let U be the subspace spanned by the $(a_{11}, a_{12}, \ldots, a_{1n}), (a_{21}, a_{22}, \ldots, a_{2n}), \ldots\ldots (a_{m1}, a_{m2}, \ldots, a_{mn})$ and let the dim V be r.

Then the system of eqn is of rank r.

~~we know that~~ $v_1 = (1, 0, \ldots, 0)$, $v_2 = (0, 1, \ldots, 0) \ldots$ $v_n = (0, 0, \ldots, 1)$ be the basis of $F^{(n)}$.

Then $\hat{v}_1, \hat{v}_2, \ldots \hat{v}_n$ be the dual basis in $\hat{F}^{(n)}$

Let $f \in \hat{F}^{(n)}$

$$\Rightarrow f = x_1 \hat{v}_1 + x_2 \hat{v}_2 + \ldots + x_n \hat{v}_n \text{ where } x_i$$

Let $f \in A(U)$. Then $f(u) = 0 \quad \forall u \in U$

$$\therefore f(a_{i1}, a_{i2}, \ldots, a_{in}) = 0$$

$$(x_1 \hat{v}_1 + x_2 \hat{v}_2 + \ldots + x_n \hat{v}_n)(a_{i1} v_1 + a_{i2} v_2 + \ldots + a_{in} v_n) = 0$$

is true for $i = 1$ to n.

$\therefore (x_1, x_2, \ldots, x_n)$ is the soln of given system

Conversely,

$(x_1, x_2, \ldots, x_n)$ is the solution of given system.

To prove: $f \in A(U)$.

ie) To prove: $f(u) = 0 \quad \forall u \in U$.

$$\text{Since } f(a_{i1}, a_{i2}, \ldots, a_{in}) = a_{i1} x_1 + a_{i2} x_2 + \ldots + a_{in} x_n = 0.$$

Let $u \in U$ then $u = \alpha_1 (a_{11}, a_{12}, \ldots, a_{1n}) + \alpha_2 (a_{21}, a_{22}, \ldots, a_{2n}) + \ldots + \alpha_m (a_{m1}, a_{m2}, \ldots, a_{mn})$

$$f(u) = f[\alpha_1 (a_{11}, a_{12}, \ldots, a_{1n}) + \alpha_2 (a_{21}, a_{22}, \ldots, a_{2n}) + \ldots + \alpha_m (a_{m1}, a_{m2}, \ldots, a_{mn})]$$

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$= \alpha_1 [f(a_{11}, a_{12}, \ldots, a_{1n})] + \alpha_2 [f(a_{21}, a_{22}, \ldots, a_{2n})] + \cdots$
$+ \alpha_m [f(a_{m1}, a_{m2}, \ldots, a_{mn})]$

$= \alpha_1 (0) + \alpha_2 (0) + \cdots + \alpha_m (0)$

$= 0$

$\therefore f(u) = 0 \ \forall u \in U.$

$\Rightarrow f \in A(U).$

Hence the number of independent solution is the system dimension of $A(U)$.

Now, $\dim A(U) = \dim F^{(n)} - \dim U$

$= n - r.$

Hence the number of linearly independent Solution is $n-r$.

Corollary

If $n > m$ ie) if the number of unknowns exceeds the number of equations then there is a solution $(x_1, x_2, \ldots x_n)$ where not all of $x_1, x_2, \ldots x_n$ are 0.

Proof: Suppose U is generated by m vectors and $m < n$, $\dim U = r \le m$.

$\therefore r \le m < n \Rightarrow n - r > 0 \Rightarrow n - r \ge 1.$

we get There is a solution $(x_1, x_2, \ldots x_n)$ in which not all of them be zero.

Sec 4.4 Inner product spaces

Defn: A vector space V over a field of real numbe is called a real vector space.

A vector space V over a field of complex numbers is called complex vector space.

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Defn:

Inner Product space

The vector space V over F is said to be an inner product space if there is defined for any two vectors $u, v \in V$ and element $\langle u, v\rangle$ in F such that

(1) $\langle u, v\rangle = \overline{\langle v, u\rangle}$

(2) $\langle u, u\rangle \geq 0$ and $\langle u, u\rangle = 0$ iff $u = 0$.

(3) $\langle \alpha u + \beta v, w\rangle = \alpha\langle u, w\rangle + \beta\langle v, w\rangle$

Defn: A function satisfying to above property is called an inner product

Note:

If F is the field of complex numbers then

(I) $\langle u, v\rangle$ is real.

(II) $\langle u, \alpha v + \beta w\rangle = \bar{\alpha}\langle u, v\rangle + \bar{\beta}\langle u, w\rangle$.

Proof:

(I) By property (i), $\langle u, v\rangle = \overline{\langle v, u\rangle}$

In particular $\langle u, u\rangle = \overline{\langle u, u\rangle}$

$\Rightarrow \langle u, u\rangle$ is real.

(II) $\langle u, \alpha v + \beta w\rangle = \overline{\langle \alpha v + \beta w, u\rangle}$ [by Property (1)]

$= \overline{\alpha\langle v, u\rangle + \beta\langle w, u\rangle}$ [by property (3)]

$= \overline{\alpha\langle v, u\rangle} + \overline{\beta\langle w, u\rangle}$

$= \bar{\alpha}\,\overline{\langle v, u\rangle} + \bar{\beta}\,\overline{\langle w, u\rangle}$

$= \bar{\alpha}\langle u, v\rangle + \bar{\beta}\langle u, w\rangle$.

Defn:

If $v \in V$ then the length of v (or norm of v), written as $\|v\|$ and is defined by $\|v\| = \sqrt{\langle v, v\rangle}$

$\sqrt{(v, v)}$ (or) $\sqrt{\langle v, v\rangle}$

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Lemma: 4.4.1

If $u, v \in V$ and $\alpha, \beta \in F$ then

$$\langle \alpha u+\beta v, \alpha u+\beta v\rangle = \alpha\bar{\alpha}\langle u,u\rangle + \bar{\alpha}\beta\langle v,u\rangle + \beta\bar{\beta}\langle v,v\rangle + \bar{\beta}\alpha\langle u,v\rangle$$

Proof:

$$\langle \alpha u+\beta v, \alpha u+\beta v\rangle = \alpha\langle u, \alpha u+\beta v\rangle + \beta\langle v, \alpha u+\beta v\rangle$$

[by property 3]

$\langle \alpha u+\beta v, w\rangle = \alpha\langle u,w\rangle + \beta\langle v,w\rangle$

$$= \alpha\left[\bar{\alpha}\langle u,u\rangle + \bar{\beta}\langle u,v\rangle\right] + \beta\left[\bar{\alpha}\langle v,u\rangle + \bar{\beta}\langle v,v\rangle\right]$$

[by Note]

$$= \alpha\bar{\alpha}\langle u,u\rangle + \alpha\bar{\beta}\langle u,v\rangle + \beta\bar{\alpha}\langle v,u\rangle + \beta\bar{\beta}\langle v,v\rangle$$

Corollary: $\|\alpha u\| = |\alpha|\,\|u\|$

Proof: $\|\alpha u\|^2 = \langle \alpha u, \alpha u\rangle$

$= \alpha\bar{\alpha}\langle u,u\rangle$ [By the above Lemma]

$= |\alpha|^2\,\|u\|^2$ $\left[\because \alpha\bar{\alpha} = |\alpha|^2,\ \langle u,u\rangle = \|u\|^2\right]$

Take square roots,

$$\|\alpha u\| = |\alpha|\,\|u\|$$

Lemma: 4.4.2

If a, b, c are real numbers such that $a>0$ and $a\lambda^2 + 2b\lambda + c \geq 0$ for all real numbers λ, then $b^2 \leq ac$.

Proof:

$+\lambda^2 \geq 0$

$$a\lambda^2 + 2b\lambda + c = \frac{a^2\lambda^2}{a} + \frac{2ba\lambda}{a} + c$$

b^2 add & subtract $= \frac{1}{a}\left[a^2\lambda^2 + 2ba\lambda + b^2 - b^2\right] + c$

$$= \frac{1}{a}\left[(a\lambda+b)^2\right] + c - \frac{b^2}{a}$$

Since $a\lambda^2 + 2b\lambda + c \geq 0$ $\forall$ real number λ,

$$\frac{1}{a}(a\lambda+b)^2 + \left(c - \frac{b^2}{a}\right) \geq 0 \quad \forall \lambda \quad \text{—— (1)}$$

In particular $\lambda = \frac{-b}{a}$

$\therefore$ ① $\Rightarrow c - \frac{b^2}{a} \geq 0.$

$$\Rightarrow \frac{ac - b^2}{a} \geq 0.$$

$$ac - b^2 \geq 0.$$

$$b^2 \leq ac.$$

This is known as ~~schwarz inequality~~.

Theorem: 4.4.1 [Schwarz inequality]

If $u, v \in V$ then $|(u, v)| \leq \|u\| \|v\|$

Proof:

Case (i)

If $\boxed{u = 0}$ then $(u,v) = 0 \Rightarrow |(u,v)| = 0.$

$\|u\| \|v\| = 0$ $\quad [\because (0, v) = (0{\cdot}0, v)$
$\quad = 0(0, v)$
$\quad = 0]$

$|(u,v)| \leq \|u\| \|v\|$ is true

Case (ii)

Suppose $\boxed{(u,v) \text{ is real and } u \neq 0.}$

For any real number λ.

$$(\lambda u + v, \lambda u + v) \geq 0$$

$$\Rightarrow \lambda \bar{\lambda} (u,u) + \lambda(u,v) + \bar{\lambda}(v,u) + (v,v) \geq 0.$$

$$\Rightarrow \lambda^2 (u,u) + \lambda(u,v) + \lambda(u,v) + (v,v) \geq 0.$$

$$\Rightarrow \lambda^2 (u,u) + 2\lambda(u,v) + (v,v) \geq 0. \quad [\because (u,v) \text{ is real}]$$

Take $a = (u,u)$ and $b = (u,v)$, $c = (v,v)$

Then a, b, c are real numbers and $a > 0$. $(\because u \neq 0)$ and

$a\lambda^2 + 2b\lambda + c \geq 0.$

By above Lemma, $b^2 \leq ac$

$$(u,v)^2 \leq (u,u) \cdot (v,v)$$

$$(u,v)^2 \leq \|u\|^2 \|v\|^2$$

$$\Rightarrow |(u,v)| \leq \|u\| \|v\|.$$

Case (iii)

If $\boxed{\alpha = (u,v) \text{ is not real then } \alpha \neq 0}$

Lemma proof write panranum

$\Rightarrow \frac{u}{\alpha}$ is exists

Now, $\left(\frac{u}{\alpha}, v\right)$ ~~is real~~

Now, $\left(\frac{u}{\alpha}, v\right) = \frac{1}{\alpha}(u,v) = \frac{(u,v)}{(u,v)} = 1$.

$\Rightarrow \left(\frac{u}{\alpha}, v\right)$ is real and $\frac{u}{\alpha} \neq 0$.

By the above case, $\left|\left(\frac{u}{\alpha}, v\right)\right| \leq \left\|\frac{u}{\alpha}\right\| \|v\|$ (by Case (i))

(1) $u = \alpha\frac{u}{\alpha}$, $(\frac{u}{\alpha}, v)$ is real, $\frac{u}{\alpha} \neq 0$
(2) (u,v) is real & $u \neq 0$
(3) $\alpha(u,v)$ is not real & $\alpha \neq 0$

ie) $1 \leq \frac{1}{|\alpha|}\|u\|\|v\|$

$\Rightarrow |\alpha| \leq \|u\|\|v\|$

$|(u,v)| \leq \|u\|\|v\|$.

Example:

1) If $V = F^{(n)}$ with $(u,v) = \alpha_1\overline{\beta_1} + \cdots + \alpha_n\overline{\beta_n}$ where $u = (\alpha_1, \ldots, \alpha_n)$ and $v = (\beta_1, \beta_2, \ldots, \beta_n)$.

Then by Schwartz inequality,

$$|\alpha_1\overline{\beta_1} + \alpha_2\overline{\beta_2} + \cdots + \alpha_n\overline{\beta_n}| \leq (\alpha_1\overline{\alpha_1} + \alpha_2\overline{\alpha_2} + \cdots + \alpha_n\overline{\alpha_n})^{1/2} \cdot (\beta_1\overline{\beta_1} + \cdots + \beta_n\overline{\beta_n})^{1/2}$$

ie) $|\alpha_1\overline{\beta_1} + \alpha_2\overline{\beta_2} + \cdots + \alpha_n\overline{\beta_n}| \leq (|\alpha_1|^2 + \cdots + |\alpha_n|^2)^{1/2}(|\beta_1|^2 + \cdots + |\beta_n|^2)^{1/2}$

Squaring we get,

$$|\alpha_1\overline{\beta_1} + \alpha_2\overline{\beta_2} + \cdots \alpha_n\overline{\beta_n}|^2 \leq (|\alpha_1|^2 + \cdots + |\alpha_n|^2)(|\beta_1|^2 + \cdots + |\beta_n|^2)$$

2) If V is the set of all continuous complex valued functions on $[0,1]$ with inner product defined by

$$(f(t), g(t)) = \int_0^1 f(t)\overline{g(t)}\,dt$$

By Schwartz inequality,

$$\left|\int_0^1 f(t)\overline{g(t)}\,dt\right|^2 \leq \left(\int_0^1 f(t)\overline{f(t)}\,dt\right)\left(\int_0^1 g(t)\overline{g(t)}\,dt\right)$$

$$= \left(\int_0^1 |f(t)|^2\,dt\right)\left(\int_0^1 |g(t)|^2\,dt\right)$$

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Defn: If $u, v \in V$ then u is said to be orthogonal to v if $\boxed{(u,v) = 0}$

Note: If u is orthogonal to v then v is othongonal to u

Proof: u is orthogonal to $v \Rightarrow (u, v) = 0$

proof verdom $\Rightarrow \overline{(v, u)} = 0$

$\Rightarrow \overline{\overline{(v, u)}} = \bar{0}$

$\Rightarrow (v, u) = 0$

$\Rightarrow$ v is othogonal to u.

Defn: If W is a subspace of V then the orthogonal complement of W is defined by

$$W^{\perp} = \{x \in V / (x, w) = 0 \quad \forall w \in W\}$$
$$= \{x \in V / (w, x) = 0 \ \forall w \in W\}$$

Lemma: 4.4.3

$W^{\perp}$ is a subspace of V.

Proof:

Let $a, b \in W^{\perp}$ and $\alpha, \beta \in F$

Then $(a, w) = 0$ and $(b, w) = 0 \quad \forall w \in W$.

To prove: $\alpha a + \beta b \in W^{\perp}$

ie) $(\alpha a + \beta b, w) = 0 \quad \forall w \in W$.

Now, $(\alpha a + \beta b, w) = \alpha (a, w) + \beta (b, w)$

$= \alpha \cdot 0 + \beta \cdot 0$

$= 0$.

$\Rightarrow \alpha a + \beta b \in W^{\perp}$

Note: $\boxed{W \cap W^{\perp} = \{0\}}$

Let $w \in W \cap W^{\perp}$ proof verdom

It must be self-orthogonal.

$\Rightarrow (w, w) = 0 \ (\because w \in W^{\perp})$

$\Rightarrow w = 0$ [since $(u, u) = 0$ then $u = 0$]

$\therefore W \cap W^{\perp} = \{0\}$.

Example: 4·4·1

In $F^{(n)}$ define, for $u=(\alpha_1,\alpha_2,\ldots,\alpha_n)$ and $v=(\beta_1,\beta_2,\ldots,\beta_n)$ then $(u,v)=\alpha_1\bar{\beta}_1+\alpha_2\bar{\beta}_2+\cdots+\alpha_n\bar{\beta}_n$. This defines an inner product on $F^{(n)}$.

1) $(u,v) = \alpha_1\bar{\beta}_1+\alpha_2\bar{\beta}_2+\cdots+\alpha_n\bar{\beta}_n$

$= \bar{\beta}_1\alpha_1+\bar{\beta}_2\alpha_2+\cdots+\bar{\beta}_n\alpha_n$

$= \overline{\bar{\alpha}_1\beta_1+\bar{\alpha}_2\beta_2+\cdots+\bar{\alpha}_n\beta_n}$ proof of random

$= \overline{(v,u)}$

2) $(u,u) = \alpha_1\bar{\alpha}_1+\alpha_2\bar{\alpha}_2+\cdots+\alpha_n\bar{\alpha}_n$

$= |\alpha_1|^2+|\alpha_2|^2+\cdots+|\alpha_n|^2 \geq 0.$

If $(u,u)=0 \Leftrightarrow |\alpha_1|^2+|\alpha_2|^2+\cdots+|\alpha_n|^2=0.$

where $\alpha_1=0,\alpha_2=0\ldots,\alpha_n=0.$

$\Leftrightarrow (\alpha_1,\alpha_2,\ldots,\alpha_n)=(0,0,\ldots,0)=0.$

3) $(\alpha u+\beta v,w) = (\alpha(\alpha_1,\alpha_2,\ldots,\alpha_n)+\beta(\beta_1,\beta_2,\cdots,\beta_n),(\vartheta_1,\vartheta_2,\ldots,\vartheta_n))$

$= (\alpha\alpha_1+\beta\beta_1)\bar{\vartheta}_1+(\alpha\alpha_2+\beta\beta_2)\bar{\vartheta}_2+\cdots+(\alpha\alpha_n+\beta\beta_n)\bar{\vartheta}_n$

$= \alpha\alpha_1\bar{\vartheta}_1+\alpha\alpha_2\bar{\vartheta}_2+\cdots+\alpha\alpha_n\bar{\vartheta}_n+\beta\beta_1\bar{\vartheta}_1+\beta\beta_2\bar{\vartheta}_2+\cdots\cdots+\beta\beta_n\bar{\vartheta}_n.$

$= \alpha(\alpha_1\bar{\vartheta}_1+\alpha_2\bar{\vartheta}_2+\cdots+\alpha_n\bar{\vartheta}_n)+\beta(\beta_1\bar{\vartheta}_1+\beta_2\bar{\vartheta}_2+\cdots+\beta_n\bar{\vartheta}_n)$

$= \alpha(u,w)+\beta(v,w)$

From ①, ②, ③ it is inner product.

Example: 4·4·2

In $F^{(2)}$ define for $u=(\alpha_1,\alpha_2)$ and $v=(\beta_1,\beta_2)$, $(u,v)=2\alpha_1\bar{\beta}_1+\alpha_1\bar{\beta}_2+\alpha_2\bar{\beta}_1+\alpha_2\bar{\beta}_2$. It is an inner product on $F^{(2)}$.

~~1) $(u,v) = 2\alpha_1\bar{\beta}_1+\alpha_1\bar{\beta}_2+\alpha_2\bar{\beta}_1+\alpha_2\bar{\beta}_2$~~

on $F^{(2)}$

Example: 4·4·3 Let V be the set of all continuous complex-valued functions on the closed unit interval $[0,1]$. If $f(t), g(t)\in V$ define $(f(t),g(t))=\int_0^1 f(t)\overline{g(t)}\,dt$

It is an inner product on V.

Defn: If the set of vectors $\{v_i\}$ in V is an orthonormal set if (i) Each v_i is of length 1 (ie) $\boxed{(v_i, v_i) = 1}$

(ii) For $i \neq j$, $\boxed{(v_i, v_j) = 0}$

Lemma: 4.4.4

If $\{v_i\}$ is an orthonormal set, then the vectors in $\{v_i\}$ are linearly independent. If $w = \alpha_1 v_1 + \cdots + \alpha_n v_n$ then $\alpha_i = (w, v_i)$ for $i = 1, 2, \ldots n$.

Proof:

Suppose $\alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_n v_n = 0$

Now, $(\alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_n v_n, v_i) = (0, v_i)$

$= 0$ —— ①

Also,

$(\alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_n v_n, v_i) = \alpha_1 (v_1, v_i) + \alpha_2 (v_2, v_i) + \cdots + \alpha_i (v_i, v_i) + \cdots + \alpha_n (v_n, v_i)$

$= \alpha_1 (0) + \alpha_2 (0) + \cdots + \alpha_i (1) + \alpha_{i+1} (0) + \cdots + \alpha_n (0)$

$= \alpha_i$ —— ② [since $\{v_i\}$'s are orthonormal set]

From ① and ②, $\alpha_i = 0$.

$\therefore \{v_1, v_2, \ldots, v_n\}$ ~~is~~ are linearly independent.

Let $w = \alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_n v_n$

$(w, v_i) = (\alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_n v_n, v_i)$

$= \alpha_1 (v_1, v_i) + \alpha_2 (v_2, v_i) + \cdots + \alpha_i (v_i, v_i) + \cdots + \alpha_n (v_n, v_i)$

$= \alpha_1 (0) + \alpha_2 (0) + \cdots + \alpha_i (1) + \cdots + \alpha_n (0)$

$= \alpha_i$

$(w, v_i) = \alpha_i$

Lemma : 4.4.5

If $\{v_1, v_2, \ldots, v_n\}$ is an orthonormal set in V and if $w \in V$, then $u = w - (w, v_1)v_1 - (w, v_2)v_2 - \ldots - (w, v_i)v_i - \ldots (w, v_n)v_n$ is orthogonal to each of $v_1, v_2, \ldots, v_n$.

Proof:

To prove: u is orthogonal to each v_i

ie) To prove: $(u, v_i) = 0$.

For any $i = 1$ to n:

$$(u, v_i) = (w - (w, v_1)v_1 - (w, v_2)v_2 - \ldots - (w, v_i)v_i - \ldots - (w, v_n)v_n, v_i)$$

$$= (w, v_i) - (w, v_1)(v_1, v_i) - (w, v_2)(v_2, v_i) - \ldots (w, v_i)(v_i, v_i) \ldots - (w, v_n)(v_n, v_i)$$

$$= (w, v_i) - (w, v_i)(v_i, v_i)$$

$$= (w, v_i) - (w, v_i)(1)$$ [since it is orthonormal set $(v_i, v_j) = 0$ if $i \neq j$, $(v_i, v_j) = 1$ if $i = j$

$$= 0.$$

Hence u is orthogonal to each v_i.

Theorem: 4.4.2 [Gram - schmidt orthogonalization process]

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Let V be a finite-dimensional inner product space; then P.T V has an orthonormal set as a basis.

Proof:

Let V be a finite-dimensional inner product space with dimension n.

Let $v_1, v_2, \ldots, v_n$ be a basis of V.

Let us construct an orthonormal set of n vectors $w_1, w_2, \ldots, w_n$ as follows

$$\text{Let } w_1 = \frac{v_1}{\|v_1\|}$$

$$\|w_1\|^2 = (w_1, w_1)$$

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$$= \left(\frac{v_1}{\|v_1\|}, \frac{v_1}{\|v_1\|} \right)$$

$$= \frac{1}{\|v_1\| \, \|v_1\|} (v_1, v_1)$$

$$= \frac{1}{\|v_1\|^2} \|v_1\|^2$$

$$= 1.$$

Let $u_2 = v_2 - (v_2, w_1) w_1$

Now, $(u_2, w_1) = (v_2 - (v_2, w_1) w_1, w_1)$

$$= (v_2, w_1) - (v_2, w_1)(w_1, w_1) \quad \left[\because \|w_1\|^2 = (w_1, w_1) = 1 \right]$$

$$= (v_2, w_1) - (v_2, w_1) \cdot 1$$

$$= 0.$$

$\therefore$ u_2 is orthogonal to w_1.

<u>Claim</u>: $u_2 \neq 0$.

Suppose $u_2 = 0$

$\Longrightarrow v_2 - (v_2, w_1) w_1 = 0$

$\Longrightarrow v_2 = (v_2, w_1) w_1$

$\Rightarrow$ v_2 is <u>scalar multiple</u> of w_1 and hence of v_1 $\left[\because w_1 = \frac{v_1}{\|v_1\|} \right]$

Which is a contradiction to $\{v_1, v_2\}$ is linearly independent

$\therefore u_2 \neq 0.$

Let $w_2 = \dfrac{u_2}{\|u_2\|}$.

Now, $(w_2, w_2) = \left(\frac{u_2}{\|u_2\|}, \frac{u_2}{\|u_2\|} \right) = 1.$ and

$$(w_2, w_1) = \left(\frac{u_2}{\|u_2\|}, w_1 \right)$$

$$= \frac{1}{\|u_2\|} (u_2, w_1)$$

$$= 0 \quad \left[\because u_2 \text{ is orthogonal to } w_1 \text{ i.e.) } (u_2, w_1) = 0 \right]$$

$\therefore \{w_1, w_2\}$ is an orthonormal set.

Let $u_3 = v_3 - (v_3, w_1)w_1 - (v_3, w_2)w_2$

Then $(u_3, w_2) = (v_3 - (v_3, w_1)w_1 - (v_3, w_2)w_2 \, , w_2)$

$= (v_3, w_2) - (v_3, w_1)(w_1, w_2) - (v_3, w_2)(w_2, w_2)$

$= (v_3, w_2) - (v_3, w_2)$

$= 0.$

$(u_3, w_1) = (v_3, w_1) - (v_3, w_1)(w_1, w_1) - (v_3, w_2)(w_2, w_1)$

$= (v_3, w_1) - (v_3, w_1)$

$= 0.$

$\therefore u_3$ is orthogonal to w_1 and w_2.

claim: $u_3 \neq 0$.

Suppose $u_3 = 0$.

$\Rightarrow v_3 - (v_3, w_1)w_1 - (v_3, w_2)w_2 = 0.$

$\Rightarrow v_3$ is a linear combination of w_1 and w_2 and hence of v_1 and v_2.

which is a contradiction to $\{v_1, v_2, v_3\}$ is linearly Independent

$\therefore u_3 \neq 0.$

Let $w_3 = \dfrac{u_3}{\|u_3\|}$

$(w_3, w_3) = \left(\dfrac{u_3}{\|u_3\|}, \dfrac{u_3}{\|u_3\|}\right) = 1.$

$(w_3, w_2) = \left(\dfrac{u_3}{\|u_3\|}, w_2\right)$

$= \dfrac{1}{\|u_3\|}(u_3, w_2)$

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$(w_3, w_1) = \left(\frac{u_3}{\|u_3\|}, w_1 \right) = \frac{1}{\|u_3\|}(u_3, w_1)$

$= 0.$

$\therefore$ $\boxed{\{w_1, w_2, w_3\} \text{ is an orthonormal}}$ at proceeding like this having found orthonormal set $\{w_1, w_2, \ldots, w_i\}$.

Let $u_{i+1} = v_{i+1} - (v_{i+1}, w_1)w_1 - (v_{i+1}, w_2)w_2 - \cdots$
$- (v_{i+1}, w_i)w_i$

For $1 \leq j \leq i$

$(u_{i+1}, w_j) = (v_{i+1} - (v_{i+1}, w_1)w_1 - (v_{i+1}, w_2)w_2 - \cdots$
$\cdots - (v_{i+1}, w_j)w_j - \cdots - (v_{i+1}, w_i)w_i, w_j)$

$= (v_{i+1}, w_j) - (v_{i+1}, w_1)(w_1, w_j) - (v_{i+1}, w_2)(w_2, w_j)$
$- \cdots - (v_{i+1}, w_j)(w_j, w_j) - \cdots - (v_{i+1}, w_i)(w_i, w_j)$

$= (v_{i+1}, w_j) - (v_{i+1}, w_j)$

$= 0$

$\therefore u_{i+1}$ is orthogonal to $w_1, w_2, \ldots, w_i$.

<u>Claim:</u> $u_{i+1} \neq 0$

suppose $u_{i+1} = 0$.

$v_{i+1} - (v_{i+1}, w_1)w_1 - (v_{i+1}, w_2)w_2 - \cdots - (v_{i+1}, w_i)w_i = 0$

$v_{i+1} = (v_{i+1}, w_1)w_1 + (v_{i+1}, w_2)w_2 + \cdots + (v_{i+1}, w_i)w_i$

$\Rightarrow v_{i+1}$ is a linear combination of $w_1, w_2, \ldots, w_i$ and hence of $v_1, v_2, \ldots, v_i$.

Which is a contraction to $\{v_1, v_2, \ldots, v_i, v_{i+1}\}$ are linearly independent.

$\therefore u_{i+1} \neq 0.$

Let $w_{i+1} = \frac{u_{i+1}}{\|u_{i+1}\|}$

$(w_{i+1}, w_{i+1}) = 1$ for $1 \leq j \leq i$

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$$(w_{i+1}, w_j) = \left(\frac{u_{i+1}}{\|u_{i+1}\|}, w_j\right)$$

$$= \frac{1}{\|u_{i+1}\|}(u_{i+1}, w_j)$$

$$= 0 \quad \forall\ j = 1 \text{ to } i$$

$\therefore \{w_1, w_2, \dots, w_{i+1}\}$ is a orthonormal set

In this way we can considered orthonormal set $\{w_1, w_2, \dots, w_n\}$.

By Lemma: 4.4.4) $\{w_1, w_2, \dots, w_n\}$ are linearly independent since $\dim V = n$, then $\{w_1, w_2, \dots, w_n\}$ be a basis of V.

Problem:

Let F be the real field and let V be the set of polynomials, in a variable x, over F of degree 2 or less.

~~Soln~~ In ~~Let~~ V define an inner product by

$$(p(x), q(x)) = \int_{-1}^{1} p(x)\, q(x)\, dx.$$ construct an orthonormal set

Soln:

we know that In $V = \{1, x, x^2\}$ is a basis.

$$v_1 = 1, \quad v_2 = x, \quad v_3 = x^2$$

$$\text{Let } w_1 = \frac{v_1}{\|v_1\|} = \frac{v_1}{\|1\|} = \frac{1}{\|1\|}$$

$$\|1\|^2 = (1, 1) = \int_{-1}^{1} 1\, dx = [x]_{-1}^{1} = 1 - (-1) = 2.$$

$$\|1\|^2 = 2 \Rightarrow \|1\| = \sqrt{2}$$

$$\therefore \boxed{w_1 = \frac{1}{\sqrt{2}}}$$

$$w_2 = \frac{u_2}{\|u_2\|} \quad \text{where } u_2 = v_2 - (v_2, w_1) w_1$$

Now, $(v_2, w_1) = \left(x, \frac{1}{\sqrt{2}}\right)$

$$= \int_{-1}^{1} x \frac{1}{\sqrt{2}} \, dx = \frac{1}{\sqrt{2}} \int_{-1}^{1} x \, dx$$

$u_2 = v_2 - (v_2, w_1) w_1$, $w_2 = \frac{u_2}{\|u_2\|}$

$$= \frac{1}{\sqrt{2}} \left[\frac{x^2}{2}\right]_{-1}^{1} = \frac{1}{\sqrt{2}} \left(\frac{1}{2} - \frac{1}{2}\right)$$

$$= 0.$$

$$u_2 = v_2 - 0 . w_1 \Rightarrow u_2 = v_2$$

$$u_2 = v_2 = x.$$

$$\|u_2\|^2 = (x, x)$$

$$= \int_{-1}^{1} x^2 \, dx = 2\int_{0}^{1} x^2 \, dx = 2\left[\frac{x^3}{3}\right]_{-1}^{1} = 2\left(\frac{1}{3} - \right.$$

$$= 2 \int_{0}^{1} x^2 \, dx = 2 \left[\frac{x^3}{3}\right]_{0}^{1}$$

$$= 2\left(\frac{1}{3} - 0\right)$$

$$= \frac{2}{3}$$

$$\|u_2\| = \sqrt{\frac{2}{3}}$$

$$\therefore w_2 = \frac{x}{\sqrt{\frac{2}{3}}} = \frac{\sqrt{3}\,x}{\sqrt{2}}$$

$$\therefore \boxed{w_2 = \frac{\sqrt{3}\,x}{\sqrt{2}}}$$

$$w_3 = \frac{u_3}{\|u_3\|} \quad \text{where } u_3 = v_3 - (v_3, w_1) w_1 - (v_3, w_2) w_2$$

$$(v_3, w_1) = \int_{-1}^{1} x^2 \frac{1}{\sqrt{2}} \, dx = \frac{1}{\sqrt{2}} \int_{-1}^{1} x^2 \, dx$$

$$= \frac{1}{\sqrt{2}}\left(\frac{x^3}{3}\right)_{-1}^{1} = \frac{1}{\sqrt{2}}\left(\frac{1}{3}+\frac{1}{3}\right) = \frac{2}{\sqrt{2}\times 3}$$

$$= \frac{\sqrt{2}}{3}$$

$$(v_3, w_2) = \int_{-1}^{1} x^2 \frac{\sqrt{3}x}{\sqrt{2}}\,dx = \frac{\sqrt{3}}{\sqrt{2}}\int_{-1}^{1} x^3\,dx$$

$$= \sqrt{3/2}\left[\frac{x^4}{4}\right]_{-1}^{1} = \sqrt{3/2} \times \left(\frac{1}{4} - 1/4\right)$$

$$(v_3, w_2) = 0$$

$$u_3 = v_3 - \frac{\sqrt{2}}{3} \times \frac{1}{\sqrt{2}} - 0$$

$$u_3 = x^2 - 1/3$$

$$\|u_3\|^2 = (u_3, u_3) = \int_{-1}^{1}\left(x^2 - 1/3\right)\left(x^2 - 1/3\right)dx$$

$$= \int_{-1}^{1}\left(x^2 - 1/3\right)^2 dx = \int_{-1}^{1}\left(x^4 - 2/3\,x^2 + \frac{1}{9}\right)dx$$

$$= \left[\frac{x^5}{5} - \frac{2x^3}{3\times 3} + \frac{1}{9}x\right]_{-1}^{1}$$

$$= \left(1/5 - \frac{2}{9} + \frac{1}{9}\right) - \left(-1/5 + 2/9 - 1/9\right)$$

$$= \frac{1}{5} - \frac{2}{9} + \frac{1}{9} + 1/5 - 2/9 + 1/9$$

$$= 2/5 - 2/9 = \frac{18-10}{45}$$

$$\|u_3\|^2 = 8/45$$

$$\|u_3\| = \frac{2\sqrt{2}}{3\sqrt{5}}$$

$$\therefore w_3 = \frac{u_3}{\|u_3\|} = \frac{x^2 - 1/3}{2\sqrt{2}/3\sqrt{5}}$$

$$= \frac{3x^2-1}{3} \times \frac{3\sqrt{5}}{2\sqrt{2}}$$

$$= \frac{(3x^2-1)\sqrt{5}}{2\sqrt{2}} \times \frac{\sqrt{2}}{\sqrt{2}} = \frac{(3x^2-1)\sqrt{10}}{4}$$

$$w_3 = \frac{(3x^2-1)\sqrt{10}}{4}$$

$\therefore \left\{ \frac{1}{\sqrt{2}}, \frac{\sqrt{3}x}{\sqrt{2}}, \frac{(3x^2-1)\sqrt{10}}{4} \right\}$ is an orthonormal set

Theorem: 4.4.3

If V is a finite-dimensional inner product space and if W is a subspace of V, then $V = W + W^{\perp}$. More particularly, V is the direct sum of W and $W^{\perp}$ (ie) $V = W \oplus W^{\perp}$

nest **Proof**: First proof:

Let V be a finite-dimensional inner product space and W be a subspace of V.

<u>To prove</u>: $V = \cancel{W \oplus W^{\perp}}$ $W + W^{\perp}$

claim: $V = W \oplus W^{\perp}$

since V is finite dimensional then W is also a finite dimensional

since V is inner product space then W is itself an inner product space.

Thus W is a finite dimensional inner product space

By Gram schmidt orthogonalization Process,

we can find an orthonormal basis of W

Let $v \in V$.

By Lemma: 4.4.5,

$v_0 = v - (v, w_1)w_1 - (v, w_2)w_2 - \cdots - (v, w_r)w_r$ is

$v = v_0 + (v, w_1)w_1 + (v, w_2)w_2 + \cdots + (v, w_r)w_r$ — (1)

a orthogonal to each of $w_1, w_2, \ldots, w_r$

since $\{w_1, w_2, \ldots, w_r\}$ is an orthonormal basis of W then v_0 is orthogonal to every element in W

ie) $(v_0, w) = 0 \ \forall\, w \in W.$

$\Rightarrow v_0 \in W^{\perp}$

By (1), $v \in W^{\perp} + W$. — (2)

$\therefore V \subseteq W^{\perp} + W$

clearly $W^{\perp} + W \subseteq V$

Thus $V = W + W^{\perp}$

By Note, $W \cap W^{\perp} = \{0\}$

Hence $V = W \oplus W^{\perp}$

Second Proof:

Assume that F is ~~the~~ field of real number

Let $v \in V$.

Suppose that we find a vector $w_0 \in W$ such that $\|v - w_0\| \leq \|v - w\| \ \forall\, w \in W$

Claim: $v - w_0 \in W^{\perp}$

ie) To Prove: $(v - w_0, w) = 0 \ \forall\, w \in W$.

Let $w \in W$ then $w + w_0 \in W$ [$\because w_0 \in W$ and V is vector space]

We our assumption, $\|v - w_0\| \leq \|v - (w_0 + w)\|$.

ie) $(v - w_0, v - w_0) \leq ((v - w_0) - w, (v - w_0) - w)$

$= (v - w_0, v - w_0) - (v - w_0, w) - (w, v - w_0) + (w, w)$

$= (v - w_0, v - w_0) - 2(v - w_0, w) + (w, w)$

$\Rightarrow 2(v - w_0, w) \leq (w, w)$ is true for $w \in W$.

In particular $\frac{w}{m} \in W$ where m is the integer

$$\Rightarrow 2\left(v - w_0, \frac{w}{m}\right) \leq \left(\frac{w}{m}, \frac{w}{m}\right)$$

$$\Rightarrow \frac{2}{m}(v - w_0, w) \leq \frac{1}{m^2}(w, w)$$

$$\Rightarrow 2(v - w_0, w) \leq \frac{1}{m}(w, w)$$

As $m \to \infty$, $\frac{1}{m}(w, w) \to 0$.

$$\therefore 2(v - w_0, w) \leq 0$$

$$\Rightarrow (v - w_0, w) \leq 0 \quad \text{--- (3)}$$

Similiarly, we prove $(v - w_0, -w) \leq 0$

ie) $-(v - w_0, w) \leq 0$.

$$\Rightarrow (v - w_0, w) \geq 0 \quad \text{--- (4)}$$

From (3) and (4)

$(v - w_0, w) = 0$ true for $w \in W$.

$$\Rightarrow v - w_0 \in W^{\perp}.$$

$$\Rightarrow v \in W^{\perp} + w_0 \subseteq W^{\perp} + W.$$

$$\Rightarrow V \subseteq W^{\perp} + W.$$

clearly $W + W^{\perp} \subseteq V$.

$$\therefore W + W^{\perp} = V.$$

By note, $W \cap W^{\perp} = \{0\}$.

hence $V = W \oplus W^{\perp}$.

Next to prove the existence of $w_0 \in W$ such that $\|v - w_0\| \leq \|v - w\| \; \forall w \in W$.

Let $u_1, u_2, \ldots, u_k$ be a basis of W.

Let $w \in W$ Then $w = \lambda_1 u_1 + \lambda_2 u_2 + \cdots + \lambda_k u_k$.

Now,

$$(v-w, v-w) = (v - \lambda_1 u_1 - \lambda_2 u_2 - \cdots \lambda_k u_k, v - \lambda_1 u_1 - \lambda_2 u_2 - \cdots - \lambda_k u_k)$$

$$= (v,v) - (v, \lambda_1 u_1) - (v, \lambda_2 u_2) - \cdots - (v, \lambda_k u_k) - (\lambda_1 u_1, v) + (\lambda_1 u_1, \lambda_1 u_1) + (\lambda_1 u_1, \lambda_2 u_2) + \cdots + (\lambda_k u_k, \lambda_k u_k) - (\lambda_2 u_2, v) + (\lambda_2 u_2, \lambda_1 u_1) + \cdots + (\lambda_2 u_2, \lambda_k u_k) - (\lambda_k u_k, v) + (\lambda_k u_k, \lambda_1 u_1) + \cdots + (\lambda_k u_k, \lambda_k u_k)$$

$$= (v,v) - 2\sum \lambda_i \vartheta_i + \sum_{i,j} \lambda_i \lambda_j \beta_{ij}$$

where $\vartheta_i = (v, u_i)$ and $\beta_{ij} = (u_i, u_j)$

This is a quadratic function in the λ's is nonnegative, then by results from the calculus has a minimum.

Choose the minimum λ's as $\lambda_1^{(0)}, \lambda_2^{(0)}, \ldots, \lambda_k^{(0)}$ then a desired vector is

$w_0 = \lambda_1^{(0)} u_1 + \lambda_2^{(0)} u_2 + \cdots + \lambda_k^{(0)} u_k$ (ie) $w_0 \leq W$.

$$\Rightarrow \|v - w_0\| \leq \|v - w\|$$

Hence $\exists\ w_0 \in W$ such that $\|v - w_0\| \leq \|v - w\|$ $\forall\ w \in W$.

Corollary:

AP-2019 If V is a finite-dimensional inner product space and W is a subspace of V then $(W^\perp)^\perp = W$.

Proof: Let $w \in W$.

To prove: $w \in (W^\perp)^\perp$

ie) To prove: $(w, u) = 0\ \forall\ u \in W^\perp$.

Let $u \in W^{\perp}$ then $(u, w) = 0$ $(\because w \in W)$

$\Rightarrow (w, u) = 0$ is true for all $u \in W^{\perp}$

$\Rightarrow w \in (W^{\perp})^{\perp}$

$\therefore W \subseteq (W^{\perp})^{\perp}$ —— ①

1st, 2nd proof write pannanum.

By the above Thm, $V = W \oplus W^{\perp}$

$\Rightarrow \dim V = \dim W + \dim W^{\perp}$ —— (*)

Also $W^{\perp}$ is a subspace of V and by the above Theorem, $V = W^{\perp} \oplus (W^{\perp})^{\perp}$

$\Rightarrow \dim V = \dim W^{\perp} \oplus \dim (W^{\perp})^{\perp}$ —— (**)

From (*), (**).

$\dim W + \dim W^{\perp} = \dim W^{\perp} + \dim (W^{\perp})^{\perp}$

$\Rightarrow \dim W = \dim (W^{\perp})^{\perp}$ —— ②

From ①, ② we get, $W = (W^{\perp})^{\perp}$.